The concurrent game semantics of Probabilistic PCF

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Abstract
We define a new games model of Probabilistic PCF (PPCF) by enriching thin concurrent games with symmetry, recently introduced by Castellan et al, with probability. This model supports two interpretations of PPCF, one sequential and one parallel. We make the case for this model by exploiting the causal structure of probabilistic concurrent strategies. First, we show that the strategies obtained from PPCF programs have a deadlock-free interaction, and therefore deduce that there is an interpretation-preserving functor from our games to the probabilistic relational model recently proved fully abstract by Ehrhard et al. It follows that our model is intensionally fully abstract. Finally, we propose a definition of probabilistic innocence and prove a finite definability result, leading to a second (independent) proof of full abstraction.

1 Introduction
What is the right setting for the denotational semantics of probabilistic programs? Numerous proposals exist. Early attempts [29, 19], in the setting of domain theory, involved the probabilistic powerdomain, with which it is notoriously difficult to obtain a satisfying cartesian closed category [20]. In 2002, Danos and Harmer [13] showed that making the model more intensional offers a much more mathematically tractable development: they construct a fully abstract games model for Probabilistic Algol, an extension of Plotkin’s PCF [27] with ground mutable state and probabilistic choice. Later on, Danos and Ehrhard gave a model of Probabilistic PCF (PPCF) in probabilistic coherence spaces [12], stemming from work on Linear Logic and quantitative semantics [16], and later proved to be fully abstract [15]. In a different direction, recently Staton et al [30, 17] (followed even more recently by Ehrhard et al [14]) introduced denotational models for probabilistic programming, with a focus on continuous distributions, not previously supported.

This variety of models for a large part extends existing semantics for deterministic programs. However, without probability, game semantics [18, 3] has offered a more modular picture, accommodating in a single framework pure functional computation along with computational effects such as state [4, 2], control [21], and many others1, following the well-known research programme pushed by Abramsky [1] under the name of semantic cube. Besides this modularity w.r.t. the available computational effects in the language, game semantics also offers tools to relate models. For instance, the standard cartesian closed category of Hyland-Ong games and innocent strategies embeds functorially in the relational model [7]. Under this time-forgetting operation, points of the relational model are understood as certain states reached by strategies, without any temporal information.

Of this nice picture however, little remains outside of the deterministic case. It is unclear how to equip Danos and Harmer’s model [13] with a notion of probabilistic innocence extending the deterministic one, and how this model relates with alternative, less intensional semantics for probabilistic programs. In fact, even the preliminary question of non-deterministic innocence was unsolved until a few years ago [9, 32], when the important conceptual step was made to switch to a framework expressing explicit branching in strategies, representing more intensional behaviour. Adding quantitative information, this suggests the possibility of pushing the semantic cube towards probabilistic computation, yielding a valuable tool in our understanding of probabilistic programs.

In this paper, we make an important step in this direction. We draw on recent developments in so-called concurrent game semantics [8], a framework for game semantics built around the idea that the causality of computation (rather than plain temporal information) is primitive. In particular, we combine the thin concurrent games with symmetry [10, 11] of Castellan et al, used to build a parallel model of PCF [10], and the probabilistic concurrent strategies of [35]. We use this to build a games model of PPCF refining [13].

To support this model, we propose two further contributions. First, we give a quantitative extension of Boudes’ theorem [7] and show that our model has a functorial collapse to the \(\mathbb{R}_+\)-weighted relational model [22]. This builds on a key lemma independent of probabilities: that the condition of visibility from [10] ensures that composition of strategies is deadlock-free, and so inherently relational-like (an important precursor for that is Melliès’ games model of Linear Logic [24]). As probabilistic coherence spaces embed faithfully in the weighted relational model, it follows by [15] that our model is intensionally fully abstract in the sense of Abramsky et al [3]. As a bonus we show that this holds both for a sequential interpretation of PPCF and for a parallel one, representing independence of sub-computations. However, definability fails.

Secondly, to get back definability we introduce a notion of sequential probabilistic innocent strategy, equivalent to standard innocent strategies in the deterministic case. Sequential probabilistic innocent strategies form a refined model of PPCF for which we prove finite definability (though only \(w.r.t.\) the sequential interpretation), yielding an independent proof of intensional full abstraction (in fact, unlike previously, inequational full abstraction holds).

Related work. Our probabilistic games are related to Tsukada and Ong’s sheaf-based notion of innocence [31], though precise connections have not been investigated. That innocent strategies compose relationally is used in Melliès’ work on game semantics for linear

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1This significant achievement led the authors of the seminal papers on game semantics to receive last year’s Alonzo Church Award for Outstanding Contributions to Logic and Computation, awarded by SIGLOG, EACSL and the Kurt Godel Society.
logic [5, 24], and exploited in Boudes’ work on relating games with
the relational model — our deadlock-free property generalises it to
a non-sequential and non-innocent setting.

Outline. In Section 2 we introduce the semantics of probabilistic
programs: we describe PPCF, its relational semantics, and the prob-
abilistic event structures used to represent it in concurrent games.
In Section 3 we develop the compositional aspects of the model,
and prove the collapse to weighted relations. Finally in Section 4,
we prove full abstraction: first as a consequence of the collapse,
then (after adding innocence) via definability.

2 Semantics for Probabilistic Programs

2.1 Probabilistic PCF

We present the language PPCF, the extension of Plotkin’s PCF
[27] with a probabilistic primitive coin : Bool. Its types are those
obtained from the basic types Bool and Nat, and the arrow ⇒. Its
terms are the following:

\[ M, N \vdash \lambda x. M \mid M \mid N \mid x \mid \text{if } M \mid N \mid \text{if } M \mid N \mid Y \]

n | pred M | succ M | iszero M | coin

The typing rules are standard and omitted – we assume that in
if M N1 N2, N1 and N2 have ground type (Bool or Nat), a general
if can be defined as syntactic sugar.

The usual call-by-name operational semantics for PCF gener-
alises to a probabilistic reduction relation \( \overset \sim \rightarrow \), for \( p \in [0,1] \). All
rules are straightforward, with the primitive coin representing a
fair coin: \( \text{coin} \overset \sim \rightarrow \frac{1}{2} \) for all \( b \in \{\text{t }, \text{f }\} \). Because reduction is non-
deterministic, there can be countably many reduction paths from
\( M \) to \( N \), i.e. sequences of the form \( M \overset p_1 \overset p_2 \overset \ldots \overset p_n \rightarrow \overset p_n \rightarrow M_n = N \).

Given such a path \( p \), its weight \( w(p) \) is \( \prod_{1 \leq i \leq n} p_i \), and we define the
coefficient \( Pr(M \rightarrow N) \) as \( \sum_{p} w(p) \mid p \) is a path from \( M \) to \( N \).

Definition 2.1. Let \( M \) and \( N \) be PPCF terms such that \( \Gamma \vdash M \rightarrow A \)
and \( \Gamma \vdash N \rightarrow A \). We write \( M \preceq_{\text{ctx}} N \) if for every context \( C[\ldots] \) such
that \( \vdash C[P] : \text{Bool} \) for every \( \Gamma \vdash P : A \),

\[ \Pr(C[M] \rightarrow b) \leq \Pr(C[N] \rightarrow b) \]

for \( b \in \{\text{t }, \text{f }\} \). The equivalence induced by this preorder, contextual
equivalence, is denoted \( \equiv_{\text{ctx}} \).

2.2 The weighted relational model

In [15], Ehrhard et al proved that probabilistic coherence spaces
(PCoh) are fully abstract for PPCF: two PPCF terms are contextu-
al equivalent iff they have the same denotation in PCoh. In fact,
PCoh is cut down (via biorthogonality) from a more liberal model
PRel, the \( \mathbb{R}_+ \)-weighted relational model [22], which we also refer to
as the probabilistic relational model.

The relational model of PCF. Ignoring probability for now, the
relational model of PCF records the input-output behaviour of a
term, along with the multiplicity of resources.

Write \( \mathbb{B} = \{\text{t }, \text{f }\} \) and \( M_{\text{t }}(X) \) for the set of finite multisets
of elements of a set \( X \). Objects of \( M_{\text{t }}(X) \) are written with square
brackets with elements annotated with their multiplicity; e.g. we have
\[ \text{[t }^2, \text{f }] \in M_{\text{t }}(\mathbb{B}), \text{ where t has multiplicity 2 and f has multiplicity 1}. \]

Using this notation, the term \( b_1 : \text{Bool}, b_2 : \text{Bool} \) if \( b_1 \) \( b_2 \) \( \text{Bool} \) will be represented as the subset of \( M_{\text{t }}(\mathbb{B}) \times M_{\text{t }}(\mathbb{B}) \times \mathbb{B} \) containing:

\[ M_{\text{t }}(\mathbb{B}) \times M_{\text{t }}(\mathbb{B}) \times \mathbb{B} \]

\[ (\text{[t }^2, \text{f }], \text{[t }^2, \text{f }], \text{[t }^2, \text{f }]) \]

The model is non-uniform: it shows how the term behaves if its
argument ever changes its mind.

The interpretation of PCF in the relational model follows the
usual methodology of denotational semantics, and in particular the
interpretation of the simply-typed \( \lambda \)-calculus in a cartesian closed
category, see e.g. [23] for an introduction. To construct the target
cartesian closed category, we start with one of the simplest models
of linear logic: the category \( \text{Rel} \) of sets and relations. In \( \text{Rel} \)
the linear logic connectives are interpreted as follows: given \( X \) and \( Y \),
\( X \otimes Y = X \rightarrow Y = X \times Y, X \& Y = X + Y \) (the tagged disjoint
union) and \( !X = M_{\text{t }}(X) \). The cartesian closed category \( \text{Rel} \) is then
the Kleisli category for the comonad \( ! \), see e.g. [26]. We omit the
details of the interpretation of PCF in \( \text{Rel} \), which we will cover in
the presence of probabilities.

The weighted relational model. Since the model is non-uniform,
it supports non-deterministic primitives. Enriching this non-uniform
model with quantitative information gives the probabilistic rela-
tional model: each element comes with a weight, as shown for instance in the interpretation of \( M_{\text{t }} = b : \text{Bool} \) if \( b \) (coin \( \text{b} \) \( \perp \)) \( (\text{if } b \text{ coin } b \perp) : \text{Bool} \), where \( \perp \) is a diverging term, e.g. \( Y (\lambda x. x) \):

\[ M_{\text{t }}(\mathbb{B}) \times \mathbb{B} \]

\[ (\text{[t }^2, \text{f }], \text{[t }^2, \text{f }], \text{[t }^2, \text{f }]) \]

The weights can be greater than 1, because a multiset may cor-
respond to several execution traces. In the example above the pair
\( (\text{[t }^2, \text{f }], \text{[t }^2, \text{f }]) \) has weight \( \frac{2}{3} = \frac{1}{2} + 1 \), summing over the different orders
in which \( b \) can take its values from \( \{\text{t }, \text{f }\} \).

The pure relational interpretation from before was based on the
category \( \text{Rel} \) with objects sets and morphisms from \( X \) to \( Y \) relations \( \varphi \subseteq X \times Y \), i.e. “matrices” \( (x,y)_{x,y \in X \& Y} \in \{0,1\}^{X \& Y} \).

Accordingly, the composition of relations can be regarded as matrix
multiplication: \( (\varphi \circ \psi)_{x,z} = \sum_{y \in Y} (\varphi_{x,y} \times \psi_{y,z}) \).

So one may construct a probabilistic variant of \( \text{Rel} \) by simply
replacing the boolean semiring \( \{0,1\}, \lor, \land \) above by the semiring
\( \mathbb{R}_+ \oplus \mathbb{R}_+ \) where \( \mathbb{R}_+ = \mathbb{R}_+ \cup \{0\} \) denotes the non-negative real
numbers, with the infinity added to ensure convergence of the
potentially infinite sum in the composition formula:

\[ (\varphi \circ \psi)_{x,z} = \sum_{y \in Y} (\varphi_{x,y} \times \psi_{y,z}), \]

for \( \varphi, \psi \in \mathbb{R}_+^{X \& Y} \), \( \psi \in \mathbb{R}_+^{Y \times X} \).

There is a category \( \text{PRel} \) with sets as objects, and as morphisms
from \( X \) to \( Y \) the matrices \( \varphi \in \mathbb{R}_+^{X \& Y} \), composed as above. The
identity on \( X \) is the diagonal matrix \( [\delta_{x_1,x_1}]_{x_1 \in X} \) where \( \delta_{x_1,x_1} \) is
1 whenever \( x_1 = x_2 \) and 0 otherwise.

Now, just like \( \text{Rel} \), \( \text{PRel} \) supports the structure of a model of
linear logic with the constructions on objects the same as in \( \text{Rel} \)
and analogous constructions on morphisms. We proceed to define
the interpretation of PPCF in \( \text{PRel} \). As for \( \text{Rel} \) the interpretation of the
\( \lambda \)-calculus combinators follows from the cartesian closed structure
of the Kleisli category \( \text{PRel} \), which we do not detail further [23].
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The interpretation of $Y$ is also obtained in a standard way as a least upper bound of finite approximations, using that homsets of $\mathbf{PRel}$ are dcpos when ordered componentwise. We now focus on the interpretation of ground types and associated combinator.

The types $\mathbf{Bool}$ and $\mathbf{Nat}$ are interpreted by the sets $\mathbb{B} = \mathbb{B}$ and $\mathbb{N} = \mathbb{N}$, respectively. For $n \in \mathbb{N}$, the constant $n$ has semantics given by $(\langle n \rangle, n + 1) \mapsto 1$. The boolean constants $\mathit{t}$ and $\mathit{ff}$ are interpreted in the same way. The semantics of $\mathit{suc}$ and $\mathit{pred}$ are defined by

\[
\begin{align*}
\mathit{suc} : & \quad M_1(\mathbb{N}) \times \mathbb{N} \to \mathbb{P}^+ \\
& ((\langle n \rangle), n + 1) \mapsto 1 \\
& ((\langle n \rangle), n) \mapsto 0 \\
\mathit{pred} : & \quad M_1(\mathbb{N}) \times \mathbb{N} \to \mathbb{P}^+ \\
& ((\langle n + 1 \rangle), n) \mapsto 1 \\
& ((\langle 0 \rangle), 0) \mapsto 1 \\
& ((\langle 0 \rangle), 1) \mapsto 0
\end{align*}
\]

The morphism $\mathit{iszero} : \mathbb{P} \in \mathbf{PRel}(\mathbb{N}, \mathbb{B})$ is defined similarly. Given terms $M : \mathbf{Boo}l, N : \mathbb{X}, P : \mathbb{X}$, where $\mathbb{X}$ denotes any ground type, i.e. $\mathbf{Boo}l$ or $\mathbf{Nat}$, the term $\mathit{if} MN P$ has semantics $(\langle M \rangle, \langle N \rangle, \langle P \rangle) \circ \mathit{if}$, where $\mathit{if} \in \mathbf{PRel}(\mathbb{B} \otimes (\langle \mathbb{X} \rangle \otimes \langle \mathbb{X} \rangle) \otimes \langle \mathbb{X} \rangle, \langle \mathbb{X} \rangle, \langle \mathbb{X} \rangle)$ is defined by

\[
\begin{align*}
\mathit{if} : & \quad M_1(\mathbb{B}) \times M_1(\langle \mathbb{X} \rangle) \times M_1(\langle \mathbb{X} \rangle) \times \langle \mathbb{X} \rangle \to \mathbb{P}^+ \\
& ((\mathit{t}), \langle x \rangle, \langle y \rangle, x) \mapsto 1 \\
& ((\mathit{ff}), \langle x \rangle, \langle y \rangle, x) \mapsto 1 \\
& \quad (\mathit{t}, \mathit{ff}, \langle x \rangle, \langle x \rangle) \mapsto 0
\end{align*}
\]

Finally, the probabilistic primitive $\mathit{coin}$ is interpreted as expected by having $\mathbb{P}^+ \mathit{coin}$ and $\mathbb{P}^+ \mathit{coin}$, completing the interpretation of PPCF.

In order to avoid infinite weights, the authors of [15] do not stop with $\mathbf{PRel}$: they cut down the category using a biorthogonality construction and obtain another weighted model of linear logic, $\mathbf{PCoh}$. In $\mathbf{PCoh}$ weights remain finite, and the interpretation of a term of ground type $M : \mathbb{X}$ yields a sub-probability distribution on $\langle \mathbb{X} \rangle$. In fact, the main result of [15] is that $\mathbf{PCoh}$ is fully abstract, i.e. for any $M, N$ we have that $M \simeq_{\text{cts}} N$ if and only if $\mathbb{P}^+ M \mathit{PCoh} = \mathbb{P}^+ N \mathit{PCoh}$.

Interestingly this entails that, despite its drawbacks, $\mathbf{PRel}$ is itself already fully abstract! Indeed there is an obvious faithful forgetful functor $\mathbf{PRel} \to \mathbf{PCoh}$ preserving all the structure on the nose — in fact a term $M$ has exactly the same interpretation in $\mathbf{PRel}$ and $\mathbf{PCoh}$, the only difference being that the latter is more informative as it carries correctness information w.r.t. biorthogonality.

Although its proof is not reproducible in $\mathbf{PRel}$, the main theorem of [15] can be stated as:

\[\textup{Theorem 2.2. For any terms } \Gamma \vdash M : A \textup{ and } \Gamma \vdash N : A \textup{ of PPCF, } M \simeq_{\text{cts}} N \iff [M]_{\mathbf{PRel}} = [N]_{\mathbf{PRel}}.\]

Accordingly, in the rest of this paper, we will work only in $\mathbf{PRel}$ and ignore biorthogonality.

2.3 Game semantics and event structures

The interpretation of a term in $\mathbf{PRel}$ “flattens out” its behaviour: it only displays the multiplicity of its use of resources, but forgets in what order these resources are evaluated. This is opposed to game semantics, which also records the order in which computational events are performed, or at least the causal dependencies between them. In the concurrent game semantics presented here (very close to [10]), the term $b : \mathbf{Boo}l \vdash M = \mathit{if} b b \mathit{ff} : \mathbf{Boo}l$ can be represented by either of the two diagrams in Figure 1 (i.e. there will be two interpretation functions, sending $M$ to one or the other).

These diagrams, read from top to bottom, represent dialogues (or collections of dialogues) between two players $\mathit{Player}$ and $\mathit{Opponent}$, respectively playing for a program and its execution environment. Nodes, called moves, are computational events. Moves are due to either $\mathit{Player}$ (+) or $\mathit{Opponent}$ (−), as indicated by their polarity, and are annotated by a Question/Answer labelling $(Q, A)$: questions correspond to variable calls, whereas answers correspond to calls returning. Wiggly lines denote incompatible branchings: moves related by them cannot occur together in an execution.

The diagram on the left is a tree, and each of its branches denotes a dialogue between Player (playing for $M$) and Opponent (playing for the environment) tracing one possible execution path of $M$. For instance, the leftmost path reads:

\[\begin{align*}
q_1 & : \mathit{Opponent} : \text{"What is the output of } M \text{ (on } \mathbf{Boo}l_1)\text{"} \\
q_2 & : \mathit{Player} : \text{"What is the value of } b \text{ (on } \mathbf{Boo}l_1)\text{"} \\
q_3 & : \mathit{Opponent} : \text{"The value of } b \text{ is } \mathit{t}\text{"} \\
q_4 & : \mathit{Player} : \text{"Then, what is, again, the value of } b\text{"} \\
q_5 & : \mathit{Opponent} : \text{"The value of } b \text{ is } \mathit{t}\text{"}
\end{align*}\]

In particular, this dialogue explicitly displays the several consecutive calls to $b$, leaving Opponent the opportunity to change his mind. The full diagram on the left-hand side of Figure 1 appends all such dialogues together in a single picture, the wiggly lines separating incompatible branches.

But beyond simple sequential execution, our framework for game semantics, as it is based on an independence model of concurrency, also supports a partial order-based representation of parallel executions. The diagram on the right-hand side of Figure 1 represents another implementation strategy for $M$. Taking advantage that the order of evaluation is irrelevant in PPCF, the diagram expresses that one can evaluate the two occurrences of $b$ in parallel. For each pair of results for the two independent calls to $b$, there is a Player answer to the original Opponent question $q_2$. Rather than just chronological contiguity, the arrows there describe the causal dependency of a move, i.e. the events that must have occurred before. We will see later that both diagrams denote (up to minor details, explained later) objects called strategies, representing terms. We will describe later two interpretations of PPCF as strategies: one sequential, one parallel, respectively computing the two strategies of Figure 1 from $M$. 

\[\text{Figure 1. Two strategies for } b : \mathbf{Boo}l_1 \vdash M = \mathit{if} b b \mathit{ff} : \mathbf{Boo}l_2.\]
Diagrams such as in Figure 1, that convey information about both causal dependency and incompatibility, are naturally formalised as event structures, a concurrent analogue of trees.

**Definition 2.3.** An event structure is \((E, \leq_E, \text{Con}_E)\) with a set \(E\) of events, \(\leq_E\) a partial order stipulating causal dependency, and \(\text{Con}_E\) a non-empty set of consistent subsets of \(E\), such that

- \([e] = \{e' \mid e' \leq e\}\) is finite for all \(e \in E\)
- \([e] \in \text{Con}_E\) for all \(e \in E\)
- \(Y \subseteq X \in \text{Con}_E \implies Y \in \text{Con}_E\)
- \(X \in \text{Con}_E\) and \(e \leq e' \in X \implies X \cup [e] \in \text{Con}_E\).

With an eye to game semantics, an event structure with polarity (esp) is an event structure \(E\) with a function \(\text{pol} : E \to \{\ast, +\}\).

**Notations.** Write \(e \twoheadrightarrow e'\) for immediate causality, i.e. \(e < e'\) with no events in between. Write \(C(E)\) for the set of finite configurations of \(E\), i.e. those finite \(x \subseteq E\) such that \(x \cup [e] \in \text{Con}_E\) and \(x\) is down-closed, i.e. if \(e \leq e' \in x\) then \(x \in E\). Configurations of the form \([e]\), i.e. with a top element, are called prime configurations.

If \(E\) has polarity, we might give information about the polarity of events by simply annotating them as in \(e^+, e^-\). If \(xy \in C(E)\), write \(x \subseteq^+ y\) (resp. \(x \subseteq^- y\)) if \(x \subseteq y\) and every event in \(y\) \(\setminus x\) has positive (resp. negative) polarity.

If for an event structure \(E\) there is a binary relation \(\#_E\) such that for all \(x \subseteq E\) finite, \(x \subseteq \text{Con}_E\) if \(\forall e \neq e' \in x, \neg(e \#_E e')\), we say that \(E\) has binary conflict. In that case we automatically have that if \(e \#_E e'\) and \(e' \leq e\) then \(e \#_E e''\) as well (the conflict is inherited). If \(e \#_E e'\) and the conflict is not inherited (meaning that for all \(e_0 < e\) and \(e'_0 < e'\) we have \(\neg(e_0 \#_E e'_0)\)), we say that \(e \#_E e'\) is a minimal conflict, written \(e \overset{\#}{\sim} e'\). With all that in place, it should now be clear how the diagrams of Figure 1 denote event structures (with binary conflict) where rather than \(\leq_E\) and \#_E, we draw immediate causality \(\to\) and minimal conflict \(\sim\).

As strategies, we will see later that the esps of Figure 1 also come with a labelling function to a game representing the typing judgment \(\text{Bool} \vdash \text{Bool}\), labelling from which the annotations \(q_2^\top (x), q_2^- (x), q_2^\bot (x), \ldots\) follow. But let us first discuss how probability is adjoined to event structures.

### 2.4 Event structures with probability

**Sequential probabilistic esps.** Sequential esps (such as that on the left of Figure 1) are those for which the causal dependency is forest-shaped, and for every configuration \(x \in C(E)\), if \(x\) has several distinct extensions \(x \cup \{e_1\}, x \cup \{e_2\} \in C(E)\) with positive events, then \(x \cup \{e_1, e_2\} \notin C(E)\). This means that for every \(x \in C(E)\), there is a set of positive extensions \(\text{ext}^+_x(x)\), all pairwise incompatible.

Sequential esps are easily enriched with probabilities, following the game semantics of Probabilistic Idealized Algol of Danos and Harmer [13]. The basic idea is that for each \(x \in C(E)\), Player equips the set of extensions \(\text{ext}^+_x(x)\) with a sub-probability distribution. But rather than having a sub-distribution for each probabilistic branching in an esp, it is more convenient to carry a single valuation

\[v : C(E) \to [0, 1]\]

putting together all the local probabilistic choices: the valuation assigned to \(x\) records all the Player probabilistic choices performed in order to reach \(x\). Because \(v\) only records Player’s probabilistic choices, it is then natural to require that (1) \(v(\emptyset) = 1\) and (2) \(v(x \cup \{e\}) \geq v(x)\) (\(e^-\)) = \(v(x)\) for any negative extension \(e^-\) of \(x\). So as to enforce that local choices give sub-probability distributions, we also have (3) for all \(x \in C(E)\),

\[v(x) = \sum_{e \in \text{ext}^+_x(x)} v(x \cup \{e\}) \geq 0\]

Furthermore, \(v\) is then entirely determined by the data of \(v(e^+)\) for all positive \(e \in E\), hence a probabilistic sequential esp can be represented by annotating positive events with the valuations of their prime configuration. Figure 2 displays the esp to be later obtained as the interpretation of the term \(M_+\) (given in 2.2), with the probabilistic valuation written on the left of events.

**General probabilistic esps.** For non-sequential esps the axioms (1) and (2) still make sense, but finding the analogue of (3) is trickier, as there may be overlap between all positive extensions. This overlap leads to a redundancy in the valuation, that has to be corrected following the inclusion-exclusion principle, as in [35].

**Definition 2.4.** A probabilistic esp consists in an esp \((E, \leq_E, \text{Con}_E, \text{pol}_E)\) and a valuation \(v : C(E) \to [0, 1]\) satisfying (1), (2) above, plus (3) if \(y \subseteq^+ x_1, \ldots, x_n\) then

\[v(y) = \sum_{i=1}^n (-1)^{i+1} v \left( \bigcup_{i \in I} x_i \right) \geq 0\]

where the sum ranges over \(\emptyset \neq I \subseteq \{1, \ldots, n\}\) s.t. \(\bigcup_{i \in I} x_i \in C(E)\).

We pointed out in the beginning of Section 2.3 that the deterministic term \(M\) can be interpreted by either esp in Figure 1 – likewise, the probabilistic term \(M_+\) can be interpreted by the probabilistic esp of Figure 2, or by some probabilistic version of an event structure much like the right hand side diagram of Figure 1. However, unlike for sequential probabilistic esps, for general ones the valuation cannot always be pushed to events and has to remain on configurations. Consider for instance how one may assign a valuation \(v\) to the esp

\[q_1^\top (x), q_2^\top (x), \ldots\]

the configurations \(q_1, q_2\) necessarily have coefficient 1. Consider then letting \(v(q_1, q_2, t_1) = v(q_1, q_2, t_2) = \frac{1}{2}\), \(v(q, t_2) = v(q_1, q_2, t_2) = \frac{1}{2}\); nothing forces \(t_1\) and \(t_2\) to be probabilistically independent events, i.e. we may have \(v(q_1, q_2, t_1, t_2)) = \frac{1}{4}\). In fact the axioms would allow any value \(0 \leq p \leq \frac{1}{4}\). The assignment \(v(q_1, q_2, t_1, t_2)) = \frac{1}{4}\), for example, would indicate a probabilistic dependence between \(t_1\) and \(t_2\).

![Figure 2](image_url)
2.5 Games and strategies as esps

So far, we have explained the formal nature of the strategies interpreting terms as (probabilistic) esps, but we have not said what games they play on.

**Arenas.** The games (arenas) will themselves be certain esps—a type \( A \) will be interpreted by a arena \([A]\), listing all the computational events existing in a call-by-name execution on this type and specifying the causality and compatibility constraints on these events. The arena will also remember the polarity of each event, and whether it is a question or an answer.

Consider the ground types Bool and Nat. There are only two events available between an execution environment and a term of ground type: the environment starting the evaluation of the term (Opponent question) and the evaluation finishing (Player answer).

Accordingly, the corresponding arenas are:

\[
\text{[Bool]} = 1_{(+,\mathbb{R})} \cdots 1_{(+,\mathbb{R})} \quad \text{[Nat]} = 1_{(+,\mathbb{R})} \cdots 1_{(+,\mathbb{R})} \cdots
\]

Again, the diagrams are read from top to bottom—immediate causality in arenas is represented by dashed lines rather than arrows, to keep it easily distinguishable from causality in strategies. Although the two notions have the same formal nature, they play a different role in the development.

In a typing judgment such as \( \text{Bool}_1 \vdash \text{Bool}_2 \) there are more computational events available: upon receiving the initial question on \( \text{Bool}_2 \), Player might interrogate \( \text{Bool}_1 \), where polarity is reversed. In fact, in our running examples \( M \) and \( M_c \). (from Figures 1 and 2), Player interrogates \( \text{Bool}_1 \) twice, showing the need to create copies of \( \text{Bool}_1 \). Accordingly, the sequent \( \text{Bool}_1 \vdash \text{Bool}_2 \) will be interpreted by the arena:

\[
\text{[Bool}_1 \vdash \text{Bool}_2]\] = 1_{(+,\mathbb{R})} \cdots 1_{(+,\mathbb{R})} \cdots
\]

Note the new annotations \( q_{i}^{(+,\mathbb{R})} \) in copies of the initial question of the argument. This copy index \( i \) is implicit in the moves \( q_{i}^{(+)} \) in Figures 1 and 2. They will be introduced formally via an exponential modality. We now give the general definition of arenas.

**Definition 2.5.** An arena consists of a esp \( A \), and a labelling function \( \lambda_A : A \to (Q, \mathcal{A}) \) such that:

- \( A \) is a forest: if \( a_1 \leq a_2 \) and \( a_2 \leq a_3 \), \( a_1 \leq a_2 \) or \( a_2 \leq a_1 \).
- \( A \) is alternating: if \( a_1 \to a_2 \) then \( \text{pol}(a_1) \neq \text{pol}(a_2) \).
- \( A \) is race-free: if \( a_1 \leadsto a_2 \) then \( \text{pol}(a_1) = \text{pol}(a_2) \).
- Questions: if \( a_1 \) is minimal or if \( a_1 \to a_2 \) then \( \lambda_A(a_1) = Q \).
- Answering is affine: for every \( a_1 \in A \in C(A) \) with \( \lambda_A(a_1) = Q \), there is at most one \( a_2 \in A \) s.t. \( a_1 \to a_2 \) and \( \lambda_A(a_2) = A \).

An arena (or esp) \( A \) is a negative if every minimal event is negative.

**Strategies.** Now that we have our notion of games, we can finish making formal the strategies displayed in Figures 1 and 2.

As pointed out earlier, the diagrams of Figure 1 have to be understood as representing esps labelled by the arena, here \([\text{Bool}_1 \vdash \text{Bool}_2]\). Modulo the (arbitrary) choice of copy indices for occurrences of \( q_{i}^{(+,\mathbb{R})} \), this labelling function is implicit in the name of nodes of the diagram. However, not all such labelled esps make sense as strategies. In order to have a well-behaved notion of strategy, we will now give a number of further constraints, best introduced in multiple stages. First, we introduce pre-strategies.

**Definition 2.6.** A (probabilistic) pre-strategy on arena \( A \) is a (probabilistic) esp \( S \) along with a labelling function \( \sigma : S \to A \) such that (1) for all \( x \in C(S) \), the direct image \( \sigma x \in C(A) \) is a configuration of the game, and (2) \( \sigma \) is locally injective: for all \( s_1, s_2 \in C(S) \) if \( \sigma s_1 = \sigma s_2 \) then \( s_1 = s_2 \).

Conditions (1) and (2) amount to the fact that the function on events \( \sigma : S \to A \) is also a map of event structures [33] from \( S \) to \( A \) (ignoring here the further structure on \( S \) and \( A \)).

Although pre-strategies give a reasonable mathematical description of concurrent processes performed under the rules of a game (or protocol) \( A \), it is too general: in particular, the current definition ignores polarity. Even in a sequential world, we expect of a definition of strategy that e.g. Player cannot constrain the behaviour of Opponent further than what is specified by the rules of the game. For our strategies on event structures, Rideau and Winskel [28] proved that we need more in order to get a category: They define:

**Definition 2.7.** A pre-strategy \( \sigma : S \to A \) is a strategy if it is:

- **receptive**: for \( x \in C(S) \), if \( \sigma x \subseteq C(A) \), there is a unique \( x \subseteq x' \in C(S) \) such that \( \sigma x' = \{ y \} \); and
- **courteous**: for \( x, s' \in S \), if \( x \leadsto s' \) and if \( \text{pol}(s) = + \) or \( \text{pol}(s') = - \), then \( \sigma x \leadsto A \sigma s' \).

Thus a strategy can only pick the positive events it wants to play, and for each of those, which Opponent moves need to occur before. It was proved in [28] and further detailed in [8] that strategies can be composed, and form a category (up to isomorphism) whose structure we will revisit in the next section, aiming for an interpretation of PPCF.

But for now we still have some definitions to give on strategies. Indeed although at this point the causal structure of strategies is sufficiently well-behaved to fit in a compositional setting, as per usual in game semantics strategies have to be restricted further to ensure that they “behave like terms of PPCF”. Typically, a set of further conditions on strategies is deemed adequate when it induces a definability result, leading to full abstraction. Here instead, our conditions will first ensure that there is a functorial collapse operation to the already fully abstract probabilistic relational model. We will add further conditions in Section 4 to prove definability.

Our conditions are a subset of those of [10]. They crucially rely on the following definition.

**Definition 2.8.** A grounded causal chain (gcc) in an esp \( S \) is a set \( \rho = \{p_1, \ldots, p_n\} \subseteq S \) such that \( p_1 \) is minimal in \( S \) and \( p_1 \leadsto p_2 \leadsto p_3 \leadsto \ldots \leadsto p_n \). Note that some \( p_i \) may have dependencies not met in \( \rho \). We write gcc(S) for the set of gccs in \( S \).

Grounded causal chains give a notion of thread in this concurrent setting. The following definition ensures that each thread can be regarded as a standalone sequential program.

**Definition 2.9.** A strategy \( \sigma : S \to A \) is visible iff for all \( \rho \in \text{gcc}(S) \), we have \( \sigma \rho \in C(A) \).

As arenas are forest-shaped, any non-minimal \( a \in A \) has a unique predecessor just(a) → A a. Likewise, by local injectivity of \( \sigma \), for any \( s \in S \) whose image is non-minimal there is a unique \( s' \in S \), its justifier, such that \( \sigma s' \leadsto A \sigma s \), which we also write to as just(s).

With that in mind, the visibility of \( \sigma : S \to A \) can be equivalently stated by asking that for all \( \rho \in \text{gcc}(S) \), for each \( p \in \rho \), we have...
just(ρ₁) ∈ ρ as well. This is reminiscent of the visibility condition in HO games, which states that the justifier of a Player move always happens within the P-view [18]. In our setting however, visibility says that a strategy can be regarded as a bag of sequential threads, sometimes forking with each other, sometimes merging, and sometimes conflicting. The strategy pictured in Figure 3 is non-visible, since the justifier of ρ₁ is absent from the gcc q₁ → Ξ₁.

Each of these sequential threads needs to respect the call-return discipline, in order to forbid strategies behaving like e.g. call/cc [21]. In a set X ⊆ S, we say that an answer x ∈ X (which is shortcut for λA(σ₁₂) = A) answers a question x₁ ∈ X iff σ₁₂ →A σ₁₁ (i.e., just(σ₁₂) = σ₁₁). If a gcc ρ ∈ gcc(S) has some unanswered questions, we say that its pending question is the latest unanswered question, i.e. the maximal unanswered question for ≤ₚ.

We import from HO games [18]:

Definition 2.10. A visible strategy σ : S → A is well-bracketed iff for all ρ = [ρ₁ → S → ... → ρₙ₊₁] ∈ gcc(S), ρₙ₊₁ answers the pending question of [ρ₁ → S ... → ρₙ].

3 Compositional Structure and Collapse

3.1 A category of games and probabilistic strategies

We start by recalling some basic constructions on esp. Given an esp A, its dual is the esp A⊥ whose events, causality and consistency are exactly those of A, but polarity is reversed: pol_A⊥(a) = −pol_A(a).

Given a family (Aᵢ)ᵢ∈I of esp, we define their simple parallel composition to have events

\[ \prod_{i∈I} Aᵢ = \coprod_{i∈I} (i) × Aᵢ \]

componentwise causal ordering and polarity. The consistent sets are the finite \( \prod_{i∈I} Xᵢ \) for \( i ∈ I \) and \( Xᵢ ∈ \text{Con}_Aᵢ \) for all \( i ∈ I \).

These constructions extend to arenas with \( λ_A = λA⊥ \) and \( λA|||Aᵢ \), defined componentwise. A (probabilistic) strategy a from A to B is a (probabilistic) strategy on \( A⊥ ||| B \). Sometimes we write \( σ : A → B \) for a strategy \( σ : S → A⊥ ||| B \), keeping the S anonymous.

We now show how to compose strategies. As usual in game semantics composition involves two steps: interaction and hiding. We will first show them without probabilities, and then add it back.

Interaction of strategies. Let A, B and C be arenas, and \( σ : S → A⊥ ||| B \) and τ : T → B⊥ ||| C be strategies. Intuitively, states of the interaction τ ⊗ σ should correspond to so-called synchronised pairs:

\[ \{(xₛ, xₜ) | σ xₛ = xₐ || xₖ & τ xₜ = xₖ || xₖ₂ \} \]

According to this, the interaction of σ of Figure 3 with either τ₁ or τ₂ from Figure 1 (regarded as strategies on \( \text{Bool}_⊥ ||| \text{Bool}_⊥ \)) would have the same maximal state

\[ \{(q₁, q₂, Ξ₁, Ξ₁₂), (q₁₂, q₁, Ξ₁, Ξ₁₂)\} \]

However this seems inaccurate, because while σ wants to play Ξ₁ after q₁₁, τ₁ will only ask q₁₂ after σ plays Ξ₁; there is a causal loop.

To get an ess whose configurations correspond to causally reachable pairs of synchronised configurations, we use the following pullback in the category of esp, which we know exists from [28, 8]:

\[ \begin{array}{ccc}
T ⊗ S & → & S \parallel C \\
\sigma & ↘ & \sigma ||| C \parallel A ⊗ T \parallel B \\
\Pi₁ & ↘ & \Pi₂ \\
A & → & T \parallel S \parallel C \\
\end{array} \]

Either path around yields the interaction \( τ ⊗ σ : T ⊗ S → A || B \parallel C \), a labelled event structure, characterised in e.g. [8]:

Lemma 3.1. Configurations of \( T ⊗ S \) are in one-to-one correspondence with the synchronised pairs

\[ \{(xₛ, xₜ) | σ xₛ = xₐ || xₖ & τ xₜ = xₖ || xₖ₂ \} \]

that are causally reachable. Formally, the induced bijection \( φ : xₛ || xₖ \rightarrow xₖ \parallel xₖ₂ \) is secured, i.e. the relation on the graph of \( φ \) generated by \( (σ, τ) = (φ, φ') \) if \( t ≤'_t s \) or \( t ≤' s ∧ i ≤' t \) is a partial order.

In the interaction of σ and τ₁ above, the state \( (q₁₁, (q₂, q₁₁)) \) is maximal. It cannot be extended further, as we have a deadlock: strategies are waiting on each other. This process of eliminating causal loops is the main difference between game semantics and relational semantics; and the reason why typically mapping game semantics to relational-like models is not functorial, as in e.g. [36]. Accordingly our main result will rely on Lemma 3.7, which states that the composition of visible strategies is always deadlock-free.

Composition of strategies. Following [28, 8], from \( τ ⊗ σ : T ⊗ S → A || B \parallel C \), we set \( T ⊗ S \) to comprise the events of \( T ⊗ S \) mapped to either A or C, with the data of an event structure inherited. Thus, each \( x ∈ C(T ⊗ S) \) has a unique witness \( \{x\}_T S \in C(T ⊗ S) \).

Polarities in \( T ⊗ S \) are set so that the restriction \( τ ⊗ σ : T ⊗ S → A || B \parallel C \) preserves them. From this we get the composition of \( σ \) and \( τ \), a strategy \( τ ⊗ σ : T ⊗ S → A⊥ ||| B \parallel C \).

Composition of probabilistic strategies. We turn to the probabilistic case. For the interaction \( T ⊗ S \), for \( x ∈ C(T ⊗ S) \) we set:

\[ v_T S(x) = v_T S(xₛ) × v_T S(xₜ) \]

where \( Π₁ x = xₛ || xₙ \) and \( Π₂ x = xₙ || xₜ \). For \( x ∈ C(T ⊗ S) \), we set \( v_T \hat{S}(x) = v_T \hat{S}(xₜ) \). From [35], we know that this makes \( τ ⊗ σ \) a probabilistic strategy. We have defined

\[ τ ⊗ σ : T ⊗ S → A⊥ ||| B \parallel C \]

to be a probabilistic strategy from A to C.

The probabilistic copycat strategy. The identity strategy on an arena A is the copycat strategy, \( σ_A : A ||| A → A⊥ ||| A \). The events, consistent subsets and polarity of \( σ_A \) are those of \( A⊥ ||| A \), with causality relation \( ≤_{σ_A} \) defined as the transitive closure of

\[ ≤_{σ_A} \cup \{(1, a), (2, a)\} \text{ and } \text{pol}_{σ_A} (1, a) = − \]

\[ \cup \{(2, a), (1, a)\} \text{ and } \text{pol}_{σ_A} (2, a) = − \].

Configurations of \( σ_A \) are certain configurations \( x₁ ||| x₂ ∈ C(A⊥ ||| A) \). Being deterministic, copycat is easily made probabilistic by assigning probability 1 to every configuration [35]. Under these definitions the map \( σ_A : C_A → A⊥ ||| A \) is a probabilistic strategy.
Equivalences of strategies. It is often not sensible to compare strategies up to strict equality; for instance the associativity and identity laws for composition only hold up to isomorphism of strategies. Let $\sigma : S \rightarrow A$ and $\tau : T \rightarrow A$ be probabilistic strategies on an arena $A$. A morphism from $\sigma$ to $\tau$ is a map of essp $f : S \rightarrow T$ such that $\tau \circ f = \sigma$, and for all $x \in C(S)$, $\nu_S(x) \leq \nu_T(f(x))$. Then $\sigma$ and $\tau$ are isomorphic if there are morphisms $f : S \rightarrow T$ and $g : T \rightarrow S$ of probabilistic strategies which are inverses as maps of essp.

 Arenas, probabilistic strategies, and morphisms between them form a bicategory [28]. We will not use the 2-cells, so in what follows we work in the induced category (obtained by quotienting homsets). We are interested in a subcategory whose morphisms are the visible, well-bracketed strategies of Section 2.5, which are moreover negative (i.e. $S$ is negative) and well-threaded (for all $s \in S$, $[s]$ has exactly one initial move). These additional conditions are needed for the categorical structure presented in the next section.

Definition 3.2. The category PG has

- objects: negative arenas;
- morphisms from $A$ to $B$: negative, well-threaded, visible and well-bracketed probabilistic strategies, up to isomorphism.

3.2 A symmetric monoidal closed category

Monoidal structure. The tensor $A \otimes B$ is simply defined as $A \| B$, with unit 1 the empty arena. From $s_1 : S_1 \rightarrow A_1^\perp \| B_1$ and $s_2 : S_2 \rightarrow A_2^\perp \| B_2$, form $s_1 \otimes s_2 : S_1 \otimes S_2 \rightarrow (A_1 \otimes A_2)^\perp \| (B_1 \otimes B_2)$, as obvious from $s_1 \otimes s_2 ; (\nu_{S_1} \otimes \nu_{S_2})(x_1 \otimes x_2) = \nu_{S_1}(x_1) \otimes \nu_{S_2}(x_2)$. Without probabilities, this yields a symmetric monoidal structure [8]; the extension with probabilities offers no difficulty.

Cartesian structure. The empty arena 1 is a terminal object. The cartesian product of arenas $A$ and $B$, written $A \times B$, has events, causality, and polarity those of $A \| B$, and consistent subsets those finite $X = X_A \| \emptyset$ with $X_A \in C_A$ or $X = \emptyset \| X_B$ with $X_B \in C_B$. We have two projections:

\[ \sigma_A : C_A \rightarrow (A \& B)^\perp \| A \quad \sigma_B : C_B \rightarrow (A \& B)^\perp \| B \]

where one component of the & is not reached — this is compatible with receptivity since $A$ and $B$ are negative. From $\sigma : S \rightarrow A \| B$ and $\tau : T \rightarrow A \| C$, their pairing

\( \sigma \otimes \tau : S \& T \rightarrow A \| B \)

is obtained from $\sigma$ and $\tau$ in the obvious way. The valuation is $\nu_{S \& T}(x) = \nu_S(x)$ and $\nu_{S \& T}(\emptyset \| x_T) = \nu_T(x_T)$. The incompatibility between $B$ and $C$ is key in ensuring local injectivity. Compatibility of pairing and projections, along with surjective pairing, are easy verifications.

Closed structure. Because our objects are negative arenas, $A \| B$ usually lies outside PG. So, inspired by the arrow construction in HO game semantics, we deviate from $A \| B$ by having $A$ depend on $\min(B)$ the minimal events of $B$. If there are several of them, we copy $A$ accordingly. As our setting is sensitive to linearity, we use consistency to ensure that this copying remains linear.

Definition 3.3. Consider $A$, $B$ two negative arenas. The arena $A \rightarrow B$ has as events $\{b \in \min(B) : A_b^\perp \| B \}$ and polarity induced. The causal order is that above, enriched with pairs $((2, b), (1, (b, a)))$ for each $b \in \min(B)$ and $a \in A$. Notice that there is a function

\[ X_{A,B} : A \rightarrow B \rightarrow A^\perp \| B \]

\( (1, (b, a)) \rightarrow (1, a) \)

\( (2, b) \rightarrow (2, b) \)

collapsing all copies. We set $\text{Con}_{A \rightarrow B}$ so as to make $X_{A,B}$ a map of essp, i.e. $\{b \in \min(X_B) \mid X_b \in \text{Con}_{A \rightarrow B} \}$ iff $X_B \in \text{Con}_B$, $\biguplus_{b \in \min(X_B)} X_b \in \text{Con}_A$, and this union is disjoint.

One may then check that there is a natural bijection $PG(A \otimes B, C) \cong PG(A, B \rightarrow C)$, i.e. PG is symmetric monoidal closed.

Depo-enrichment. To interpret PCCF it will be necessary for PG to be "depo-enriched." We equip the set of probabilistic strategies on a game $A$ with a relation $\subseteq$, as follows. For $\sigma : S \rightarrow A$ and $\tau : T \rightarrow A$ probabilistic strategies, set $\sigma \subseteq \tau$ if $S \subseteq T$, i.e. $S \subseteq T$ and the structure of $S$ is the restriction of that of $T$, and if moreover $\nu_S(x) \leq \nu_T(x)$ for any $x \in C(S)$. It is clear that $\subseteq$ is a partial order. The least upper bound ( lub) of a directed set of probabilistic strategies is their union, with valuation given as $\nu(x) = \sup \nu_S(x) \mid (\sigma : S \rightarrow A) \in D$ and $x \in C(S)$. The least element (up to isomorphism) is given by $\perp_A : \min(A) \rightarrow A$ (note that the map $\emptyset \rightarrow A$ is not receptive in general and so not a strategy).

3.3 Collapsing games and strategies

Though we have yet to introduce a linear exponential comonad on PG to break linearity, we find it better to delay its introduction, and give now the collapse of arenas and strategies to sets and relations. Its functoriality will be addressed in the next subsection.

Mapping arenas to sets. Unlike games, PRel only records the trace of the data returned by functions for successful executions. In games, the relevant information is captured by the complete configurations, i.e. those $x$ where every question is answered in $x$.

Definition 3.4. Let $A$ be an arena. Define $\downarrow A$ to be the set of nonempty, complete configurations of $A$.

Consider for instance the arena $[\text{Bool}]_{\text{PG}}$ for booleans. It has two nonempty and complete configurations, $\{q, q^*\}$ and $\{q^{-}, q^{+}\}$, so $\{[\text{Bool}]_{\text{PG}}$ is isomorphic to the two-element set $\{\text{true}, \text{false}\}$. Mapping strategies to matrices. Let $\sigma : S \rightarrow A$ be a (negative, well-threaded, visible, well-bracketed) probabilistic strategy. Our goal is to define a "vector" $\downarrow \sigma \in \mathbb{R}^A_+$ indexed by the nonempty and complete configurations of $A$.

Given $x \in \downarrow A$, the coefficient $\downarrow \sigma \downarrow x$ intuitively sums the probability coefficients of all the ways one can play $x$ in $S$. This is formalised using the notion of witness:

Definition 3.5. Let $\sigma : S \rightarrow A$ be a strategy and $x \in C(A)$. A witness for $x$ in $\sigma$ is $z \in C(S)$ such that $\sigma z = x$, and such that all maximal moves of $z$ have positive polarity (we say $z$ is +covered). Write $\text{wit}_S(x)$ for the set of all witnesses of $x$ in $S$.

The requirement that witnesses should not have negative maximal moves is illustrated by the following strategy on the game $\mathbb{B} \rightarrow B$, where Player calls its argument and returns immediately:

\[ q^\uparrow \times \]

\[ q^{-}\]
When flattening out this strategy, we must not include $\{t,t\}$ as a possible execution, as this would cause functoriality to fail.

We can finally define the action of $\downarrow \sigma$ on strategies.

**Definition 3.6.** Let $\sigma : S \to A$ be a (negative, well-threaded, visible, well-bracketed) probabilistic strategy. For $x \in \downarrow A$, we let:

$$(\downarrow \sigma)_x = \sum_{z \in w_{\downarrow \sigma}(x)} w_\sigma(z).$$

### 3.4 Functionality of the collapse

Following the above a morphism $\sigma : S \to A^+ \parallel B$ in $\+	ext{PG}$ collapses to a vector $\downarrow \sigma$ indexed by elements of $\downarrow (A^+ \parallel B)$. This is not quite in $\+	ext{PRel}(\downarrow A, \downarrow B)$, which would instead be indexed by elements of $\downarrow A \times \downarrow B$, i.e. pairs of nonempty configurations. For $x \parallel y \in C(A^+ \parallel B)$ to be nonempty it is enough for only one of $x, y$ to be nonempty.

And indeed $\sigma$ might output a value without inspecting its argument: there may be witnesses to $0 \parallel y \in \sigma$, so $(\downarrow \sigma)_y$ may be non-zero. However because $A, B$ and $\sigma$ are negative, there can be no witnesses for $x \parallel 0 \in \sigma$, and the coefficient $(\downarrow \sigma)_x$ is always zero.

These observations follow from $\+	ext{PG}$ being affine, whereas $\+	ext{PRel}$ is linear: a strategy can ignore its argument — and so can a morphism in the Kleisli category $\+	ext{PRel}$, but not in $\+	ext{PRel}$. Thus the target of our collapse functor will not be $\+	ext{PRel}$ but an affine version of it introduced below. Later, moving on to the cartesian closed category $\+	ext{PG}$, we will recover the usual relational model $\+	ext{PRel}$ of $\+	ext{PFC}$.

We first describe the affine version of $\+	ext{PRel}$ and its relationship with $\+	ext{PRel}$. After that, we prove functoriality of the collapse.

**The affine relational model.** Following [26, §8.10] and decompose the ↓↓! of $\+	ext{PRel}$ into a weakening modality $\downarrow$, and a duplication modality $\downarrow$, each a comonad on $\+	ext{PRel}$. For any set $X, \downarrow X$ contains its nonempty finite multisets: $\downarrow X = M_{\text{fin}}(X)$, while $\uparrow X$ has the set $X$ along with the empty multiset: $\uparrow X = X + \{[]\}$. We omit details of their structure, induced from those of ↓↓! (found e.g. in [15]).

The Kleisli category $\+	ext{PRel}$ is now a model of affine logic, with structure defined in terms of the structure of $\+	ext{PRel}$:

- **Products**: the same as in $\+	ext{PRel}$, $X \times Y = X + Y$.
- **Monoidal structure**: $X \otimes w = X \otimes Y \otimes X + Y$, with unit $\emptyset$.
- **Closed structure**: $X \multimap Y = \downarrow X \rightarrow Y$.
- **Exponential modality**: the comonad $\downarrow$ lifted to $\+	ext{PRel}$.

Lifting the comonad $\downarrow$ to $\+	ext{PRel}$ exploits a distributive law $\downarrow \rightarrow \downarrow w$, and the Kleisli category $(\+	ext{PRel}, \downarrow)$ is isomorphic to $\+	ext{PRel}$. With this in place, the collapse will be a functor:

$$\downarrow : \+	ext{PG} \to \+	ext{PRel}$$

preserving the structure required for the interpretation.

We can now define the action of the ↓↓! on a strategy $\sigma : S \to A^+ \parallel B$. For $x \in \downarrow \downarrow \sigma$, we set $(\downarrow \sigma)_y := (\downarrow \sigma)_y$, and $\downarrow \sigma_{x,y}$ as $(\downarrow \sigma)_y$. We will now check that it is a functor, leaving the preservation of further structure for later.

**A functor.** Consider $\tau : T \to B^+ \parallel C$. To show the functoriality of $\downarrow$ we must relate $(\tau \circ \sigma)$ to the Kleisli composition $\downarrow \tau \circ \downarrow \sigma$. For $x \in \downarrow \sigma$, and $y \in \downarrow C$, the latter is given as:

$$(\downarrow \tau \circ \downarrow \sigma)_x = \delta_{\downarrow \tau}(\downarrow \sigma)_y, \sum_{y \in \downarrow B} (\downarrow \sigma)_{x,y}.\downarrow \tau(y,z),$$

To show $\downarrow (\tau \circ \sigma)_x = (\downarrow \tau \circ \downarrow \sigma)_x$, we use a bijection between:

1. witnesses $w$ for $x \parallel y$ in $\tau \circ \sigma$, and
2. pairs of $(w_x, w_T)$, where $w_x$ is a witness for $x \parallel y$ in $\sigma$, and $w_T$ for $y \parallel z$ in $\tau$, for some $y \in \downarrow B$, which satisfies $\downarrow T \circ \downarrow \sigma(w) = \downarrow \sigma(w_x) \times \downarrow \tau(w_T)$. There are subtleties in both directions — the proofs are provided in Appendix B.

From (2) to (1). This direction is the most subtle, as it bumps against the reason why traditionally operations from dynamic to static semantics are only lax functorial. Indeed, recall from Lemma 3.1 that configurations of the interaction $T \oplus S$ correspond to synchronised pairs $(w_x, w_T)$ for which the induced bijection is secured. This is in contrast with (2), where witnesses are synchronised with no securedness condition. The following crucial lemma states that, when composing visible strategies, securedness is redundant.

**Lemma 3.7 (Deadlock-free lemma).** Let $x_S \in C(S)$ and $x_T \in C(T)$ such that $\sigma x_S = x_A \parallel x_B$ and $\tau x_T = x_B \parallel x_C$. Then the induced bijection $\varphi : x_S \parallel x_C \sim x_A \parallel x_T$ is secured.

So, composing visible strategies is inherently relational, from which the direction from (2) to (1) is direct.

From (1) to (2). This direction is easier: given a witness $w$ for $x \parallel y$ in $\tau \circ \sigma$, its down-closure $[w]$ in $C(T \oplus S)$ satisfies $(\tau \circ \sigma)[w] = x \parallel y \parallel z$ for some $y \in C(B)$. It may look like we are done: writing $\Pi_1[w] = w_x \parallel z$ and $\Pi_2[w] = x \parallel y \parallel w_T$ we obtain a pair $(w_x, w_T)$ of witnesses for $x \parallel y$ and $y \parallel z$. But it remains to check that $y \in \downarrow B$, i.e. that it is complete. Well-bracketing ensures this.

**Lemma 3.8.** If $w \in \text{wit}_S(x \parallel z)$, for well-bracketed visible strategies $\sigma$ and $\tau$, where $x$ and $z$ are complete, then the unique $y \in C(B)$ such that $\tau(y \parallel \sigma)[w] = x \parallel y \parallel z$ is also complete.

**Summing up.** That this is bijective follows from $+$-coveredness of the witnesses; and the required equality is obtained by summing up on both sides following this bijection. The collapse preserves identities: for any arena $A$, $\downarrow \alpha_A$ is the Kleisli identity $\downarrow\downarrow \alpha_A \to \downarrow\downarrow \alpha_A$ (i.e. the counit for $\downarrow\downarrow \alpha$).

**Theorem 3.9.** $\downarrow : \+	ext{PG} \to \+	ext{PRel}$ is a functor.

**Preservation of structure.** This functor is well-behaved. One can easily check that it preserves the order structure on morphisms: if $\sigma \leq \tau$ then $\downarrow \sigma \leq \downarrow \tau$, and furthermore $\downarrow (\vee_{\sigma \in \downarrow \tau} \sigma) = \downarrow (\vee_{\sigma \in \downarrow \tau} \sigma)$ for any directed set $\downarrow D$ — so in fact $\downarrow \downarrow \tau$ is itself dcpo-enriched. It behaves well also with respect to the categorical structure:

**Lemma 3.10.** We have the natural isomorphisms in $\+	ext{PRel}$:

$$\downarrow (A \& B) \cong \downarrow A \& \downarrow B \quad \downarrow (A \parallel B) \cong \downarrow A \oplus \downarrow B$$

Moreover, whenever $B$ has a unique initial move, we additionally have $\downarrow (A \& B) \cong \downarrow A \&_{\text{w.c.}} \downarrow B$. All associated structural morphisms are also preserved by the collapse.

## 3.5 Games and strategies with symmetry

In Section 2.5 we hinted at the need for moves to be duplicated, and adjoined copy indices. The necessity of expressing uniformity w.r.t. copy indices (see [11]) requires us to enrich our probabilistic games with a notion of symmetry.

**Probabilistic thin concurrent games.** Event structures with symmetry, introduced in [34], were applied to games in [9] and refined in [10]. For lack of space we omit details and give an informal description. The technical development can be found in Appendix A.
Our category is a probabilistic enrichment of the thin concurrent games of [10]. The objects are $\sim$-arenas, consisting of an arena $A$ and (among others) a set $A$ of bijections $\theta : x \mapsto y$ between configurations $x, y \in C(A)$, expressing that $x$ and $y$ are interchangeable, i.e. the same up to copy indices. This is subject to further axioms [11], and informs an equivalence relation on $C(A)$. Likewise, probabilistic $\sim$-strategies are $\sigma : S \rightarrow A$ where $A$ also has an isomorphism family preserved by $\sigma$, with the requirement that symmetric configurations should be assigned the same probability.

Unlike PG, this category now supports a linear exponential comonad $!$, whose Kleisli category is, as usual, a ccc:

**Lemma 3.11.** There is a cartesian closed category $\mathcal{P} \mathcal{G}_1$ having

- objects: negative $\sim$-arenas;
- morphisms $A \rightarrow B$: (negative, well-threaded, visible, well-bracketed) probabilistic $\sim$-strategies $\sigma : S \rightarrow A$.

**Interpretation of PPCF.** The interpretation of ground types as $\sim$-arenas was given in Section 2.5. It is extended to all types by setting $[A \Rightarrow B] = [[A]] \rightarrow [[B]]$. As a cartesian closed category, $\mathcal{P} \mathcal{G}_1$ supports the interpretation of the simply-typed $\lambda$-calculus [23]; as usual, a typed term $\Gamma \vdash M : B$ with $\Gamma = x_1 : A_1, \ldots, x_n : A_n$, is interpreted as a morphism:

$$[M] : ![ \bigcup_{1 \leq i \leq n} [A_i] ] \rightarrow [[B]]$$

It remains to interpret the primitives of PPCF. From $\Gamma \vdash M : \text{Bool}$, $\Gamma \vdash N_1 : \text{Bool}$, $\Gamma \vdash N_2 : \text{Bool}$, we define $[\text{if}(M N_1 N_2)]$ via composition with a deterministic $\sim$-strategy $\text{if} : [[\text{Bool}]] \& [[\text{Bool}]] \& [[\text{Bool}]] \rightarrow [[\text{Bool}]]$. There are in fact two possibilities for $\text{if}$. As in Figure 1, one is sequential and compatible with the usual interpretation of $\text{if}$ in game semantics, while the other is the parallel strategy from [10]. We omit the specific diagrams, hoping that they are easy to generalize from those of Figure 1. We denote the sequential and parallel interpretation by $[[\_]]^s$ and $[[\_]]^p$, respectively, and simply use $[[\_]]$ when the choice does not matter: in particular, both $\sim$-strategies will collapse to the same weighted relation.

Finally constants are interpreted as in the following examples:

$\text{Bool} \Rightarrow \text{Bool}$

$[[\text{true}]] = q_{(-,Q)}$

$[[\text{false}]] = q_{(-,Q)}$

$[[\text{true} \& A]] = \frac{1}{2}[[\text{true}]] + \frac{1}{2}[[A]]$

$[[\text{false} \& A]] = \frac{1}{2}[[\text{false}]] + \frac{1}{2}[[A]]$

where configurations have probability 1 unless specified otherwise. For each $\sim$-arena $A$, there is a (deterministic) fixpoint combinator $Y_A$ on $([![A] \rightarrow ![A]])^\ast \| ![A]$ allowing us to interpret $Y$ as the lub of a set of approximants, see [10] for details.

**Relational collapse.** The new subtlety in extending our functor $\downarrow : \mathcal{P} \mathcal{G} \rightarrow \text{PRel}$, from Section 3.4 is that moves in $!A$ mention specific copy indices, while finite multisets $M_f(A)$ only count multiplicity.

To address that, we refine $\downarrow A$ as the set of $=\downarrow$-equivalence classes of non-empty and complete configurations of $A$ (and similarly for $\downarrow \sigma$). The developments of Sections 3.3 and 3.4 adapt smoothly to the new framework, and we now have $\downarrow (\downarrow A) \cong M_f(\downarrow A)$.

Thus, $\downarrow$ takes $\sim$-strategies on $A$ to $\downarrow \sigma$ in $\text{PRel}(\downarrow A, !A, l B)$, which is iso $\text{PRel}(\downarrow A, !A, l B) \cong \text{PRel}(\downarrow A, l B)$. Hence we can lift it:

**Lemma 3.12.** There is a functor $\downarrow : \mathcal{P} \mathcal{G}_1 \rightarrow \text{PRel}$. It is a straightforward verification that there is an isomorphism $\theta_A : ![A]_{\mathcal{P} \mathcal{G}} \cong ![A]_{\mathcal{P} \mathcal{G}_1}$ for any type $A$ of PPCF. Moreover the functor preserves the interpretation of all PPCF primitives, so that:

**Theorem 3.13.** For any PPCF term $\Gamma \vdash M : A$,

$$\downarrow \Gamma \vdash M : \mathcal{P} \mathcal{G} = \downarrow \Gamma \vdash M : \mathcal{P} \mathcal{G}_1 = \downarrow \Gamma \vdash M : \text{PRel}.$$

**Definition 4.4.** A strategy $\sigma : S \rightarrow A$ is sequential innocent if

- for every $x \in C(S), \nu(x) \neq 0$;
Theorem 4.8. (Finite definability) Theorem 4.7

References


A Games with Symmetry

A.1 Symmetry in event structures

We first review the basics of event structures with symmetry [34], presented here as in [10] via isomorphism families.

Definition A.1. An isomorphism family on an event structure $E$ is a set $\tilde{E}$ of bijections $\theta : x \equiv y$, where $x, y \in C(E)$, s.t.

1. For all $x \in C(E)$, $i_{x_{\tilde{E}}} : x \equiv x$ in $\tilde{E}$.
2. If $\theta : x \equiv y \in \tilde{E}$ then $\theta^{-1} : y \equiv x \in \tilde{E}$.
3. If $\theta : x \equiv y$ and $\eta : y \equiv z \in \tilde{E}$ then $\eta \circ \theta : x \equiv z \in \tilde{E}$.
4. If $\theta : x \equiv y \in \tilde{E}$ and $x \not\subseteq x' \in C(E)$, then there exists $y' \equiv y \in \tilde{E}$ and $\theta' : x' \equiv y' \in \tilde{E}$ such that $\theta \subseteq \theta'$.
5. If $\theta : x \equiv y \in \tilde{E}$ and $x' \subseteq x \in C(E)$, then there exists $y' \equiv y \in \tilde{E}$ and $\theta' : x' \equiv y' \in \tilde{E}$ such that $\theta' \subseteq \theta$.

An event structure with symmetry (ess) is a pair $E = (E, \tilde{E})$ where $\tilde{E}$ is an isomorphism family on $E$. If $E$ additionally has polarities, then the bijections in $\tilde{E}$ are furthermore required to preserve them; $E$ is then an essp.

Conditions (1), (2) and (3) give $\tilde{E}$ a groupoid structure, while (4) and (5) ensure that symmetric configurations have bisimilar future and isomorphic past. We regard bijections as sets of pairs, justifying the notation $\theta \subseteq \theta'$ (or $\subseteq^+$ and $\subseteq^-$ if $E$ has polarities). If $E$ and $F$ are ess, a map of ess $f : E \rightarrow F$ preserves symmetry if for every $\theta : x \equiv y \in \tilde{E}$ (shorthand for $\theta : x \equiv y \in \tilde{E}$), the bijection $f\theta = (f(e), f(e'))$ holds in $\tilde{F}$; we write $f : E \rightarrow \tilde{F}$.

Symmetry and probability can be combined:

Definition A.2. A probabilistic essp is an essp $(E, \tilde{E})$ and a valuation $v$ on $E$ such that $v(x) = v(y)$ whenever $\theta : x \equiv y$.

In other words, symmetric configurations of a probabilistic essp must have the same probability valuation.

A.2 Thin concurrent games

We use $A, B, S, F, \ldots$ to denote essps, keeping the underlying event structures $(A, B, \ldots)$ and isomorphism families $(A, B, \ldots)$ implicit.

The construction on games introducing symmetry, and which drives the notion of essps, is the exponential $!A$. It is a symmetric, infinitary form of parallel composition:

Definition A.3. Given a family $A_i, i \in I$ of essps, their parallel composition $||_{i \in I} A_i$ is $||_{i \in I} A_i$ equipped with the isomorphism family $||_{i \in I} A_i$, with bijections $\theta : ||_{i \in I} x_i \equiv ||_{i \in I} y_i$ induced by a family $((\theta_i : x_i \equiv y_i)_{i \in I}$ such that for all $(i, a) \in ||_{i \in I} x_i$, $\theta((i, a)) = (i, \theta_i a_i)$.

Definition A.4. Let $A$ be a negative essp, i.e. $A$ is negative. Then, $!A$ is defined as $||_{i \in \omega} A$, with isomorphism family enriched to comprise the bijections $\theta : ||_{i \in \omega} x_i \equiv ||_{i \in \omega} y_i$ such that there exists a permutation $\pi : I \equiv J$ and a family $(\theta_j : A_i)_{i \in I}$, with $\theta((i, a)) = (\pi(i), \theta_j a_j)$ for all $(i, a) \in ||_{i \in \omega} x_i$.

This is very similar to the equivalence relation on the game $!A$ in AM games [3], and was also considered in [9]. Note that this ! operation is not the same as the one used in [10] and which duplicates all moves of the game “in depth” rather than just at the surface – in the spirit of HO games [18]. We prefer here this “surface” version, which allows an easier connection with the relational model as both cartesian closed categories are then obtained as Kleisli categories.

Very soon, strategies will be considered up to the choice of copy indices. But this is naively not preserved under composition – for it to be a congruence, strategies also have to be uniform: the behaviour of a strategy should not depend on the copy indices used by Opponent, although his choice of copy indices will. Constructing a framework of concurrent games where “being the same up to copy indices” is a congruence is quite challenging, see e.g. [11] for a discussion. One solution, used in [9], is to ask that all strategies are saturated, and play non-deterministically all possible copy indices. Another, introduced in [10] and detailed in [11], requires instead that strategies pick copy indices deterministically (are thin, see Definition A.6). For thin strategies to behave well we also must constrain the games, and separate Player permutations and Opponent permutations, in a way that is very reminiscent of Melliés’ notion of uniformity [25] by bi-invariance under the action of two groups of Opponent and Player permutations.

Definition A.5. A thin concurrent game (tcg) is $A = (A_\perp, \tilde{A}_\perp)$ where $A$ is an essp, and $A_\perp$ and $A_\perp$ are isomorphism families on $A$ included in $A$, such that:

1. $\theta \in A_\perp$ then $\theta = \text{id}_x$ for some $x \in C(A)$.
2. $\theta \in A_\perp$ and $\theta \subseteq^+ \theta'$ in $A$ then $\theta' \in A_\perp$.
3. $\theta \in A_\perp$ and $\theta \subseteq^- \theta'$ in $A$ then $\theta' \in A_\perp$.

When $A$ is a negative tcg, Opponent is responsible for the first layer of symmetry in $!A$: the family $!A_\perp$ comprises all $\theta : x \equiv y$ such that for all $i \in \omega, \theta_i : x_i \equiv y_{\pi(i)} \in \tilde{A}_\perp$. On the other hand the family $!\tilde{A}_\perp$ comprises all $\theta : x \equiv y$ such that for all $i \in I, \pi(i) = i$ and $\theta_i \in \tilde{A}_\perp$.

While the dual definition could also be given for positive $A$, candidates of $!A_\perp$ and $!\tilde{A}_\perp$ for $A$ with minimal events of mixed polarities inevitably fail some axioms of tcgs (and their intended consequences) – building an exponential without any assumption on polarity requires saturation [6, 9].

We now add probability to the uniform strategies of [10, 11], called $\sim$-strategies.

Definition A.6. A probabilistic $\sim$-strategy on a tcg $A$ is a map of essps $\sigma : S \rightarrow A$ (where $A = (A_\perp, \tilde{A}_\perp)$, for now) such that $S$ is a probabilistic essp, $\sigma : S \rightarrow A$ a strategy, and:

1. $\sigma$ is strong-receptive: if $\theta \in \tilde{S}$ and $\sigma\theta \subseteq^+ \eta$ in $A$, then there exists a unique $\theta \subseteq \theta' \in S$ such that $\sigma\theta' = \eta$.
2. $S$ is thin: for $\theta : x \equiv y$ s.t. $x' = x \cup \{s\} \in C(S)$ with $\text{pol}(s) = \bot$, there is a unique $t \in S \text{ s.t. } \theta \cup \{t, s\} \in \tilde{S}$.

The remaining concepts of Section 2.3 extend in the presence of symmetry: a $\sim$-arena is a tcg $A$ with a $Q$/$\mathcal{A}$ labelling $\lambda$ on $A$, such that $(A, \lambda)$ is an arena and every bijection in $A$ preserves the action of $\lambda$. $A$ $\sim$-strategy $\sigma : S \rightarrow A$ on a $\sim$-arena $A$ is visible (resp. well-bracketed) when the underlying strategy $S \rightarrow A$ is visible (resp. well-bracketed).

A.3 A category of probabilistic $\sim$-strategies

The parallel composition of tcgs $A = (A_\perp, \tilde{A}_\perp, A_{\perp})$ and $B = (B_\perp, \tilde{B}_\perp, B_{\perp})$ is $A || B = (A || B_\perp, \tilde{A}_\perp || B_{\perp}, \tilde{A}_{\perp} || B_{\perp})$. The dual of $A$ is the ess $A'^{-}$ = $(A'^{-}, \tilde{A}_\perp, A_{\perp})$. As usual, a probabilistic $\sim$-strategy from $A$ to $B$ is one on $A' \parallel B$. 
A.3.1 Composition and copycat

Given \( \sim \)-strategies \( \sigma : S \to \mathcal{A}^\perp \parallel B \) and \( \tau : T \to \mathcal{B}^\perp \parallel C \), their interaction \( T \otimes S \) is (just like its underlying event structure \( T \circ S \))
defined as the pullback

\[
\begin{array}{c}
\begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {$S \parallel C$};
\node (B) at (2,0) {$\mathcal{A} \parallel T$};
\node (C) at (0,-1) {$\sigma \| \mathcal{B} \parallel C$};
\node (D) at (2,-1) {$\tau \| T$};
\path (A) edge (B);
\path (B) edge (D);
\path (A) edge (C);
\path (D) edge (C);
\end{tikzpicture}
\end{array}
\end{array}
\end{array}
\]

only this time in the category of event structures with symmetry.

To define the composition of \( \sigma \) and \( \tau \) we equip the event structure \( T \circ S \) (obtained from \( T \otimes S \) after hiding) with the isomorphism family \( T \circ S \rightarrow C \), set to comprise the bijections \( \theta : x \cong y \) such that \( x \in C(T \circ S) \) such that \( \theta \subseteq \theta' \) for \( \theta' : [x]:T \circ S \cong C \). From this we get the composition of \( \sigma \) and \( \tau \), a \( \sim \)-strategy \( \tau \circ \sigma : T \circ S \to \mathcal{A}^\perp \parallel C \).

When \( \sigma \) and \( \tau \) are probabilistic \( \sim \)-strategies, the symmetry does not affect the definition of \( T \circ S \parallel C \) which is easily shown to be invariant under the bijections in \( T \circ S \), making it a probabilistic \( \sim \)-strategy.

Finally, the copycat strategy on a \( \sim \)-arena \( \mathcal{A} \) is also equipped with symmetry: the isomorphism family \( \mathcal{C} \parallel \mathcal{A} \) comprises all \( \theta = \theta_1 \parallel \theta_2 \parallel x_1 \parallel x_2 \parallel y_1 \parallel y_2 \) such that \( \theta_1, \theta_2 \in \mathcal{A} \) and such that \( \theta \) is an order-isomorphism.

We can form a bicategory of tcgs, probabilistic \( \sim \)-strategies, and morphisms (where a morphism of \( \sim \)-strategies is one between the underlying strategies which additionally preserves symmetry). But isomorphisms do not exploit symmetry, and distinguishing strategies playing the same moves up to copy indices. We aim for a weaker notion of isomorphism of \( \sim \)-strategy, which we will use to quotient our bicategory.

A.3.2 Weak isomorphism

**Definition A.7.** Two maps \( f, g : S \to \mathcal{A} \) of ess are symmetric, written \( f \sim g \), if for all \( x \in \mathcal{C}(S) \), the bijection \( \theta_x : \{(fx, gs) \mid s \in x\} \) is in \( \mathcal{A} \). If moreover \( A \) is a t cg, say \( f \) and \( g \) are positively symmetric, written \( f \sim^+ g \), if \( \theta_x \in A \) for all \( x \).

A weak morphism of probabilistic \( \sim \)-strategies from \( \sigma : S \to \mathcal{A} \) to \( \tau : T \to \mathcal{B} \) is a map of ess \( f : S \to T \) such that \( \sigma \circ f \sim^+ \tau \) and such that for all \( x \in \mathcal{C}(S) \), \( v_S(x) \leq v_T(f(x)) \). The induced notion of weak isomorphism yields a weaker notion of equivalence between \( \sim \)-strategies which we use to quotient our bicategory. A key result of [10, 11] is that weak isomorphism is preserved under composition, which crucially depends on the thinness axiom for \( \sim \)-strategies.

The conditions on strategies introduced in Sections 2 and 3 do not rely on the affine nature of PG. They extend directly to the framework with symmetry, so that:

**Definition A.8.** There is a category \( \mathcal{PG} \) having objects: negative \( \sim \)-arenas;

- morphisms: positive, well-threaded, visible and well-bracketed probabilistic \( \sim \)-strategies on \( \mathcal{A}^\perp \parallel \mathcal{B} \).

A.3.3 A model of intuitionistic linear logic

The category \( \mathcal{PG} \) is symmetric monoidal closed and cartesian, with all structure induced from that of PG in the obvious way (see [11]) for details). In this setting however, we can define a linear exponential comonad !.

Given a \( \sim \)-arena \( \mathcal{A} \), the esp !\( \mathcal{A} \) was defined earlier, in Definition A.4. We now define the positive and negative isomorphism families. When \( \mathcal{A} \) is a negative \( \sim \)-arena, Opponent is responsible for the first layer of symmetry in !\( \mathcal{A} \); the family !\( \mathcal{A}_- \) comprises all \( \theta : x \cong y \) such that for all \( i \in \omega \), \( \theta_i : x_i \cong y_i \in !\mathcal{A} \). On the other hand the family !\( \mathcal{A}_+ \) comprises all \( \theta : x \cong y \) such that for all \( i \in I \), \( \pi_i = i \) and \( \theta_i \in !\mathcal{A}_+ \).

The action of ! on morphisms is as follows: from \( \sigma : S \to \mathcal{A}^\perp \parallel B \) we define

\[
!\sigma : !S \to (!\mathcal{A})^\perp \parallel !B
\]

as the obvious map (easily checked to satisfy the conditions for a \( \sim \)-strategy), with probability valuation given by

\[
v_{!S}(x_i) = \prod_{i \in I} v_S(x_i)
\]

yielding a probabilistic \( \sim \)-strategy !\( \sigma \) from !\( \mathcal{A} \) to !\( B \). This construction yields a functor ! : \( \mathcal{PG} \to \mathcal{PG} \).

By adjoining deterministic \( \sim \)-strategies corresponding to the standard copycat strategies of AJM games, ! has a comonad structure \( (!, \delta, \varepsilon) \) satisfying the Seely axioms [26], turning \( \mathcal{PG} \) into a model of ILL.

B Omitted Proofs

B.1 Proof of the deadlock-free lemma (Lemma 3.7)

The key property of visible strategies that we use to prove this result is the following lemma:

**Lemma B.1.** Let \( \sigma : S \to A \) be a visible strategy and let \( s < s' \) be events of \( S \). Then the justifier of \( s' \) is comparable to \( s \).

**Proof.** Since \( s < s' \), there exists a gcc \( \rho \) of \( S \) such that \( s \) and \( s' \) occur in \( \rho \). By visibility of \( \sigma \), \( \text{just}(s') \) occurs in \( \rho \). Since \( \rho \) is a total-order, \( \text{just}(s') \) must be comparable to \( s \).

We first prove the lemma for dual visible strategies, on a game \( A \) with only negative minimal events. So consider visible \( \sigma : S \to A \) (necessarily negative), and \( \tau : T \to A^\perp \) (necessarily non-negative). We assume moreover that events in \( S \) (resp. \( T \)) that map to minimal events of \( A \) are minimal.

In such a situation, we have:

**Lemma B.2.** In a situation as above, for any \( x \in \mathcal{C}(S), y \in \mathcal{C}(T) \) such that \( \sigma x = \tau y \), the bijection \( \varphi : x \cong \sigma x = \tau y \) is induced by local injectivity, is secured.

**Proof.** Observe first that because \( \sigma x = \tau y \), it follows that \( \varphi \) preserves justifier: \( \text{just}(\varphi(s)) = \text{just}(\varphi(s)) \). We recall that \( \varphi \) is secured when the relation \( (s, t) \sim \varphi((s', t')) \) defined on graph of \( \varphi \) as \( s < s' \) or \( t < T t' \) is acyclic. Suppose it is not, and consider a cycle \( ((s_1, t_1), \ldots, (s_n, t_n)) \) with

\[
(s_1, t_1) \sim \varphi(s_2, t_2) \sim \varphi \cdots \varphi(s_n, t_n) \sim \varphi(s_1, t_1)
\]

Let us first give a measure on such cycles. The length of a cycle as above is \( n \). For \( a \in A \), the depth \( \text{depth}(a) \) of \( a \) is the length of the path to a minimal event of the arena – so the depth of a minimal event is 0. Then, the depth of the cycle above is the sum:

\[
d = \sum_{1 \leq i \leq n} \text{depth}(\sigma s_i)
\]
Cycles are well-ordered by the lexicographic ordering on \((n, d)\); let us now consider a cycle which is minimal for this well-order.

Note: in this proof, all arithmetic computations on indices are done modulo \(n\) (the length of the cycle).

Since \(s\leq_{S}\) and \(s\leq_{T}\) are transitive we can assume that \(s_{2k}\leq_{S}s_{2k+1}\)
and \(t_{2k+1}\leq_{T}t_{2k+2}\) for all \(k\). But then it follows by minimality that \(pol(s_{2k}) = -\) and \(\varphi_{2}(s_{2k+1}) = +\) so that the cycle is alternating.

Indeed, assume
\[
(s_{2k+1}, t_{2k+1}) \cdot \varphi(s_{2k+1}, t_{2k+1}) \cdot \varphi(s_{2k+2}, t_{2k+2}) \cdot \varphi(s_{2k+3}, t_{2k+3})
\]
with \(t_{2k+1} \leq_{T} s_{2k+2} \leq_{S} s_{2k+3}\). The causal dependency
\(t_{2k+1} \leq_{T} t_{2k+2}\) decomposes into \(t_{2k+1} \leq_{T} t_{2k+2}\) by

\[
\text{with \(s_{i} \leq_{S} s_{j}\) for \(i < j\) as well, therefore we can replace the cycle fragment above with}
\]

\[
(s_{2k+1}, t_{2k+1}) \cdot \varphi(s_{2k+1}, t_{2k+1}) \cdot \varphi(s_{2k+2}, t_{2k+2}) \cdot \varphi(s_{2k+3}, t_{2k+3})
\]

which has the same length but smaller depth. Absent by the dual reasoning, events with odd index must have polarity as \((s_{2k+1}, t_{2k+1})\) as well.

Now, we remark that the cycle cannot contain events that are minimal in the game. Indeed, by hypothesis a synchronised event \(s_{i}, t_{i}\) with \(s_{i} <_{S} t_{i}\) is minimal in \(A\) is such that \(s <_{S} s_{i}\) and \(t <_{T} t_{i}\) are minimal as well, so \((s, t)\) is a root for \(\varphi_{s}\) and cannot be in a cycle. Therefore, all events in the cycle have a predecessor in the game, \(i.e.\) a justifier.

Since \(s_{2k} <_{S} s_{2k+1}\), by Lemma B.1, just\((s_{2k+1})\) is comparable with \(s_{2k}\) in \(S\). They have to be distinct, as otherwise we would have \(s_{2k} \leftarrow s_{2k+1}\) which in turn implies \(t_{2k} <_{T} t_{2k+1}\). This gives \(t_{2k-1} <_{T} t_{2k+1}\) hence \((s_{k}, t_{k})\) and \((s_{k+1}, t_{k+1})\) can be removed without breaking the cycle, contradicting its minimality. By a similar reasoning, just\((t_{2k+2})\) is comparable and distinct from \(t_{2k+1}\).

Assume that we have \(s_{2k} <_{S} s_{2k+1}\) for some \(k\). Since just\((s_{2k+1}) <_{T} t_{2k+1}\) and just\((t_{2k+2}) <_{T} t_{2k+1}\), therefore, we can replace the cycle fragment
\[
(s_{2k}, t_{2k}) \cdot \varphi(s_{2k}, t_{2k}) \cdot \varphi(s_{2k+1}, t_{2k+1}) \cdot \varphi(s_{2k+2}, t_{2k+2})
\]
with the cycle fragment
\[
(s_{2k}, t_{2k}) \cdot \varphi(s_{2k+1}, t_{2k+1}) \cdot \varphi(s_{2k+2}, t_{2k+2})
\]
which has the same length but smaller depth. Absent by the dual reasoning, events with odd index must have polarity as \((s_{2k+1}, t_{2k+1})\) as well.

So we have for all \(k\), just\((s_{2k+1}) <_{S} s_{2k}\) with \(pol\((s_{2k}) = -\)
\(\text{By courtesy and the fact that \(A\) is alternating, this has factor as}
\]

\[
\text{just\((s_{2k+1}) <_{S} s_{2k}\) which is minimal, therefore, the cycle}
\]

\[
\text{is not negative. That means that we can replace the}
\]

full cycle
\[
(s_{1}, t_{1}) \cdot \varphi(s_{2}, t_{2}) \cdot \varphi(s_{3}, t_{3}) \cdot \varphi(s_{4}, t_{4}) \cdot \varphi(s_{5}, t_{5})
\]
with the cycle
\[
\text{just\((s_{1})\) and \(\varphi(s_{2}), \varphi(s_{3}), \varphi(s_{4}), \varphi(s_{5})\) which has the same length but smaller depth. Absent.}
\]

The lemma above is the core of the proof. However, some more bureaucratic reasoning is necessary to reduce Lemma 3.7, which does not talk of two dual visible strategies on one arena of fixed polarity, to the one above.

Consider \(s: A \to A^\perp \parallel B\) and \(\tau: T \to T^\perp \parallel C\) which are both visible, well-threaded negative strategies with \(A, B\) and \(C\) negative arenas. We cannot use transparently the lemma above, because the interaction of \(\sigma\) and \(\tau\) involves the closed interaction of \(\sigma \parallel C^\perp: S \parallel C^\perp \to A^\perp \parallel B\) \(\parallel C^\perp\) and \(A\parallel T \to A\parallel T^\perp \parallel C\), and the arena \(A \parallel B\parallel C\) is not negative.

Instead, we will use that the same interaction can be replayed in the arena with enriched causality \((A \hookrightarrow B) \o C\). Remark that as in Definition 3.3, we have a map:
\[
\chi_{A, B, C}: ((A \hookrightarrow B) \o C) \to A \parallel B^\perp \parallel C
\]

Using the fact that \(\sigma\) and \(\tau\) are well-threaded, these additional causal links in the games are compatible with the interaction:

**Lemma B.3.** Let \(x_{S} \in C(S)\) and \(x_{T} \in C(T)\) such that \(\sigma x_{S} \equiv x_{A} \parallel x_{B}\) \(\parallel x_{T}\) \(= x_{C}\), and consider the induced bijection (not yet known to be secured):
\[
\varphi: x_{S} \parallel x_{C} \cong x_{A} \parallel x_{T}
\]

Then, there is \(w \in C((A \hookrightarrow B) \o C)\) such that \(\chi_{A, B, C} w = x_{A} \parallel x_{B} \parallel x_{C}\) and the induced bijections:
\[
x_{S} \parallel x_{C} \cong w \parallel x_{A} \parallel x_{T} \cong w
\]
are secured.

**Proof.** By well-threadedness, each \(t \in x_{T}\) mapping to \(B\) has a unique minimal causal dependency mapping to \(C\), informing the copy of \(A \hookrightarrow B\), hence the event of \((A \hookrightarrow B) \o C\) it should be sent to. Likewise, each \(s \in x_{S}\) has a unique minimal causal dependency \(s' \in S\) mapping to \(B\), and there is some synchronisation \(((1, s'), (2, t'))\) where \(t'\) in turn has a unique minimal causal dependency mapping to \(C\) - this informs the event of \((A \hookrightarrow B) \o C\) that \(s\) should be sent to.

Securedness is immediate from the observation that the only immediate causal links added have the form \(c \to b\) or \(b \to a\) for \(a, b, c\) minimal respectively in \(A, B, C\); in both cases spanning a parallel composition in \(\parallel C\) or \(A \parallel T\).

We now need to modify \(\sigma \parallel C^\perp\) and \(A\parallel T\) so that they are dual playing on \((A \hookrightarrow B) \o C\) and \((A \hookrightarrow B) \o C\) respectively. We do that via the following two pullbacks:
\[
\begin{array}{c}
A^\perp \parallel C^\perp \\
\downarrow \quad \downarrow \quad \downarrow \\
A \parallel C \\
\end{array}
\]

\[
\begin{array}{c}
A \parallel T \\
\downarrow \quad \downarrow \\
A^\perp \parallel C \\
\end{array}
\]

One can see \(\sigma': S' \to ((A \hookrightarrow B) \o C)^{\perp}\) and \(\tau': T' \to (A \hookrightarrow B) \o C\) similarly as \(\sigma \parallel C\) and \(A \parallel T\), but with the added causality as in \((A \hookrightarrow B) \o C\), so that the games \(C, B, A\) are opened in that order. We have:

**Lemma B.4.** So defined, \(\sigma\) and \(\tau\) satisfy the conditions of Lemma B.2, i.e. they are visible and events mapping to minimal events of \((A \hookrightarrow B) \o C\) are minimal.

**Proof.** Immediate from standard arguments on the analysis of immediate causality in a pullback, see e.g. [8].
We can finally wrap up:

**Lemma 3.7.** Let $x_S \in C(S)$ and $x_T \in C(T)$ such that $\sigma x_S = x_A \parallel x_B$ and $\tau x_T = x_B \parallel x_C$. Then, the induced bijection $x_S \parallel x_C \cong x_A \parallel x_T$ is secured.

**Proof.** By Lemma B.3, we get $w \in C((A \rightarrow B) \rightarrow C)$, and pairing $w$ and $x_S \parallel x_C \parallel x_T$ (resp. $w$ and $x_A \parallel x_T$), along with the securedness property from Lemma B.3, gives us $x_S' \in C(S')$ (resp. $x_T' \in C(T')$) such that $\sigma' x_S' = \tau' x_T'$. By Lemma B.2, the induced bijection $x_S' \cong x_T'$ is secured. But this entails that the composite bijection $x_S \parallel x_C \cong x_A \parallel x_T$ is secured as well, as the constraints are weaker. \qed

**B.2 Proof of Lemma 3.8**

**Lemma 3.8.** If $w \in C(T \circ S)$ is a witness for $x \parallel z$ in the composition of well-bracketed visible strategies $\sigma$ and $\tau$, where $x$ and $z$ are complete, then the unique $y \in C(B)$ such that $(\tau \circ \sigma)[w] = x \parallel y \parallel z$ is also complete.

**Proof.** If $y$ is empty, then $y = [] \in \perp \downarrow B$ is the witness. Otherwise, we must show that $y$ is complete, so that its symmetry class $y \in \downarrow B$. So let $q \in [w]$ be a question mapped to $y$ by $\tau \circ \sigma$. Let $e$ be a visible event (we call visible events those of $w$) such that $e < e'$, chosen so that no event between $q$ and $e$ is visible. Then, by courtesy, $\text{pol}(e) = +$ since there is a causal link to $e$ in $w$ which is not in the game. By assumption, maximal events of $w$ are visible positive answers, so there is a gcc

$$\rho : \cdots \rightarrow q \rightarrow \cdots \rightarrow e^+ \rightarrow \cdots \rightarrow a^+.$$

We claim that $\rho$ and $a$ can be chosen in such a way that $\rho[e, a]$ contains no visible negative questions. In order to show this we give a construction process, which necessarily terminates since all gccs are finite. Start the process at $p_1 = e$. If $e$ is a positive answer then we are done. If it is a positive question, then by receptivity there is a visible answer $a^*$ such that $e \rightarrow a'$, so let $p_{i+1} = a'$ and continue. If it is a negative answer then it is not maximal by assumption ($w$ is $+$-covered), so continue down any gcc; the next move is either a positive visible event, and we can apply the steps above, or it is a hidden event of $[w]$, in which case we continue down any gcc until reaching a visible event (recall that there are no hidden maximal moves) and repeat the procedure.

So we have defined $\rho : \cdots \rightarrow q \rightarrow \cdots \rightarrow e^+ \rightarrow \cdots \rightarrow a^+$. By visibility, $a^+$ points to some negative $q'$ in $\rho$, which is necessarily visible since $(\tau \circ \sigma)a$ and $(\tau \circ \sigma)q'$ are in the same component. Therefore, by construction of $\rho$, $q'$ must occur before $q$ in $\rho$. By well-bracketing of $\tau$ and $\sigma$ (which implies the well-bracketing of gccs in $T \circ S$), all questions of $\rho[q', a^+]$ must be answered in $\rho[q', a^+]$, including $q$. So in particular $q$ has an answer in $w$, and $y$ is complete. \qed

**B.3 Proof of intensional full abstraction via collapse (Theorem 4.3)**

**Theorem 4.3** (Intensional full abstraction). Let $M$ and $N$ be PPCF terms such that $\Gamma \vdash M : A$ and $\Gamma \vdash N : A$. Then $M \equiv_{\text{ctx}} N$ if and only if $[\Gamma \vdash M]_{PG} \equiv_{\text{ctx}} [\Gamma \vdash N]_{PG}$.

**Proof.** (Only if). By the full abstraction result in $\text{PRel}$, $M \equiv N$ implies $[\Gamma \vdash M]_{\text{PRel}} \equiv [\Gamma \vdash N]_{\text{PRel}}$, which by Theorem 3.13 is the same as saying that $[\Gamma \vdash M]_{PG} \equiv [\Gamma \vdash N]_{PG}$. Suppose there exists $\alpha : ([\Gamma \Rightarrow A]) \Rightarrow B$ such that $\alpha \circ \Lambda([\Gamma \vdash M]_{PG}) \neq \alpha \circ \Lambda([\Gamma \vdash N]_{PG})$. This implies in particular that $[\Gamma \vdash M]_{PG} \not\equiv [\Gamma \vdash N]_{PG}$. Because $\downarrow$ is a structure-preserving functor, this is equivalent to $\not\equiv \alpha \circ \Lambda([\Gamma \vdash M]_{PG}) \not\equiv \alpha \circ \Lambda([\Gamma \vdash N]_{PG})$, a contradiction since $[\Gamma \vdash M]_{PG} \equiv [\Gamma \vdash N]_{PG}$. So no such $\alpha$ can exist, and $[\Gamma \vdash M]_{PG} \equiv_{\text{ctx}} [\Gamma \vdash N]_{PG}$.

(If). Suppose now that $[\Gamma \vdash M]_{PG} \equiv_{\text{ctx}} [\Gamma \vdash N]_{PG}$. Let $C[]$ be a context such that $C[M]$ and $C[N]$ are closed terms of type $\text{Bool}$. Then $[\text{C}[\cdot]]_{PG}$ is a probabilistic $\sim$-strategy $([\Gamma] \Rightarrow [\text{A}]) \Rightarrow B$, and therefore $[\text{C}[M]]_{PG} \equiv [\text{C}[N]]_{PG}$ since $[\Gamma \vdash M]_{PG}$ and $[\Gamma \vdash N]_{PG}$ are observationally equivalent. By adequacy (Theorem 4.1), we have $\text{Pr}([C[M] \rightarrow b]) = \text{Pr}([C[N] \rightarrow b])$ for all $b$. So $M \equiv N$. \qed

**C Comparing equational theories**

In this final section, we compare the different equational theories induced on terms of PPCF by the sequential interpretation, the parallel interpretation, and the interpretation in $\text{PRel}$.

If $\Gamma \vdash M, N : A$ are terms of PPCF, we introduce three notions of equivalences between them. We write $M \equiv_{\text{PRel}} N$ if $[\Gamma \vdash M]_{\text{PRel}} \equiv [\Gamma \vdash N]_{\text{PRel}}$, likewise we write $M \equiv_{\text{PG}} N$ if $[\Gamma \vdash M]_{\text{PG}} \equiv [\Gamma \vdash N]_{\text{PG}}$, and finally $M \equiv_{\text{N}} N$ if $[\Gamma \vdash M]_{\text{N}} \equiv [\Gamma \vdash N]_{\text{N}}$. The three induced equational theories on terms of PPCF are ordered as follows:

$$\equiv_{\text{PRel}} \supseteq \equiv_{\text{PG}} \supseteq \equiv_{\text{N}}$$

where we emphasize that the inclusions are strict, and that $\equiv_{\text{PG}}$ and $\equiv_{\text{N}}$ are incomparable. The non-strict inclusions are immediate consequences of the collapse functor.

$\equiv_{\text{PG}} \subseteq \equiv_{\text{PRel}}$. Observe the two following terms.

$M_1 = \begin{cases} x \text{ if } y \text{ then } t \text{ else } \perp \\ \text{else } \perp \end{cases}$

$M_2 = \begin{cases} y \text{ if } y \text{ then } t \text{ else } \perp \\ \text{if } x \text{ then } t \text{ else } \perp \\ \text{else } \perp \end{cases}$

These are equal in $\equiv_{\text{PRel}}$ (and $\equiv_{\text{PG}}$), but not in $\equiv_{\text{PG}}$, where we observe the evaluation order.

$\equiv_{\text{PRel}} \subseteq \equiv_{\text{PRel}}$. Technically it suffices to observe that the two terms if $\text{coin}$ then $\text{t}$ else $\text{t}$ and $\text{t}$ have a distinct interpretation as $\text{P}G$ will remember the nondeterministic branching and represent the former with two conflicting events. However, although it is not introduced in the paper, there is a simple equivalence (called "rigid image equivalence") that eliminates idempotent probabilistic choice and
would assimilate these two in $\mathcal{PG}$ as well. So more convincingly, we propose the more robust example below.

\[
M_3 = \begin{cases} 
\text{if coin} \\
\text{then} \\
\text{if } x \text{ then } \# \text{ else } \& \\
\text{else} \\
\text{if } x \text{ then } \& \text{ else } \#
\end{cases}
\]

\[
M_4 = \begin{cases} 
\text{if } x \text{ then } \text{coin} \\
\text{else} \\
\text{coin}
\end{cases}
\]

Those two terms are equal in \textsc{Prel}, but are distinguished by $\equiv_{\text{Prel}}$ where for $M_3$ we observe two immediate parallel calls to $x$, in contrast with $M_4$ where there is only one.

\[
\equiv_{\mathcal{PG}} \equiv_{\mathcal{PRel}}. \text{ Indeed, } M_1 \equiv_{\mathcal{PG}} M_2 \text{ but } M_1 \not\equiv_{\mathcal{PG}} M_2 .
\]

\[
\equiv_{\mathcal{PG}} \subseteq \equiv_{\mathcal{PRel}}. \text{ This comes from } \equiv_{\mathcal{PRel}} \text{ not being "sensible": it observes part of the computation that will never be used. More precisely, the term}
\]

\[
M_5 = \begin{cases} 
\text{if } \perp \text{ then } x \text{ else } \perp
\end{cases}
\]

satisfies $M_5 \equiv_{\mathcal{PG}} \perp$, but $M_5 \not\equiv_{\mathcal{PG}} \perp$: following the parallel interpretation we see the call to $x$ performed "speculatively", though of course it will yield no result.